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Small Deviations from Local Equilibrium for a Process Which Exhibits Hydrodynamical Behavior. II

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The symmetric simple exclusion process on $\mathbb{Z}$ with sources at $\pm L$, $L \in \mathbb{N}$ is considered. The stationary measure μ_L is studied in the limit as L diverges. The first order correction to its limit is proven to be of order $1/L$ and it is explicitly computed. The result is in agreement with the analysis of the model from the hydrodynamical point of view initiated in Ref. 1.

KEY WORDS: Hydrodynamical behavior of microscopic systems; stochastic dynamics; simple exclusion process; local equilibrium; Fourier law.

1. INTRODUCTION

The analysis of the small deviations from local equilibrium for the symmetric simple exclusion process was initiated in Ref. 1. In this paper we complete it by studying the "stationary case." We refer to Ref. 1 for motivations and notation. The problem and its solution are stated in Section 2, the proof in Section 3. The main technique is based on the introduction of the "weak coupling" between the independent and the simple exclusion processes, which might be interesting per se; cf. Section 3.1.

2. RESULTS

The symmetric simple exclusion process on $\mathbb{Z}$ with sources at $\pm L$ has been studied in Refs. 6 and 5. It is a Markov process with state space

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$\{0,1\}^{2L+1}$ and generator $\mathfrak{L}^L$ given by

$$
\begin{aligned}
(\mathfrak{L}^L f)(\eta) = (1/2) \sum_{x=-L}^{L-1} \left[f(\eta(x,x+1)) - f(\eta) \right] &+ (1/2) \sum_{\epsilon=\pm 1} \sum_{\delta=0,1} \\
\times \left\{ \delta p(\epsilon) \left[f(\eta(\epsilon L, \delta)) - f(\eta) \right] + (1-\delta) \left[1 - p(\epsilon) \right] \right. & \\
\left. \times \left[f(\eta(\epsilon L, \delta)) - f(\eta) \right] \right\} & \qquad (2.1)
\end{aligned}
$$

$$
\begin{aligned}
\eta = \eta(x), \qquad x \in [-L, L], \qquad & \eta(x) \in \{0,1\} \\
[\eta(x,x+1)](y) = \eta(y) \qquad & \text{for} \quad y \neq x, x+1 \\
= x \qquad & \text{for} \quad y = x+1 \\
= x+1 \qquad & \text{for} \quad y = x \\
[\eta(L,\delta)](y) = \eta(y) \qquad & \text{for} \quad y < L \\
= \delta \qquad & \text{for} \quad y = L \\
[\eta(-L,\delta)](y) = \eta(y) \qquad & \text{for} \quad y > -L \\
= \delta \qquad & \text{for} \quad y = -L \\
0 \leqslant p(\epsilon) \leqslant 1 \qquad & \text{for} \quad \epsilon = \pm 1
\end{aligned}
$$

The first term on the right-hand side of Eq. (2.1) describes the usual simple exclusion process, namely, each particle after an independent Poisson time of mean 1 jumps on one of its nearest-neighbor sites (with probability $\frac{1}{2}$). If the chosen place is occupied the other particle is forced to make the opposite jump. (We assume that particles are indistinguishable so that the above can be rephrased by saying that the jump is forbidden if the chosen site is occupied, point hard core condition.) The other terms on the right-hand side of Eq. (2.1) describes death and birth processes at $\pm L$. After an independent Poisson time of mean 1 a particle is created at L $[-L]$ with probability $p(1)$ $[p(-1)]$ or destroyed with complementary probability. The choice of the same intensity makes computations simpler; more general cases can, however, be handled with the same techniques. It is easy to see that, given L, $p(1)$, $p(-1)$ there is a unique stationary measure μ_L; cf. Ref. 5. From a hydrodynamical point of view the interest is focused on the local structure of μ_L as L diverges; in particular our problem is the computation of the first-order correction to the limiting Bernoulli state. We obtain the following:

Theorem 2.1. Let $\xi \in (-1, 1)$ and Λ be a bounded region. Let

$$
p(\xi) = \tfrac{1}{2}[p(1) - p(-1)]\xi + \tfrac{1}{2}[p(1) + p(-1)] \qquad (2.2)
$$

and

$$U_\xi(\eta_{\Lambda+[\xi L]}) = \sum_{x\in\Lambda} \alpha x\left[\eta(x+[\xi L]) - p\right] + \frac{1}{2}\zeta \sum_{\substack{x\neq y\\ x,y\in\Lambda}} \left[\eta(x+[\xi L]) - p\right]\left[\eta(y+[\xi L]) - p\right] \quad (2.3)$$

where

$$\alpha = (p-p^2)^{-1}p' \qquad \zeta = \tfrac{1}{2}(p-p^2)^{-2}(1-\xi^2)p'^2$$
$$p = p(\xi) \qquad p' = \tfrac{1}{2}\left[p(1) - p(-1)\right]$$

Let $\xi_L = 1/(L+1)[\xi L]$, then

$$\lim_{L\to\infty} L\left|\mu_L(\eta_{\Lambda+[\xi L]}) - \nu_{p(\xi_L)}(\eta_{\Lambda+[\xi L]})\exp\left\{\frac{1}{L}U_\xi(\eta_{\Lambda+[\xi L]})\right\}\right| = 0 \quad (2.4)$$

were if ρ is a probability measure on $\{0,1\}^{\mathbb{Z}}$

$$\rho(\eta_{\Lambda+[\xi L]}) = \rho\left(\{\eta(x) = 1, \forall x\in\Lambda+[\xi L]\}\right)$$

and ν_p is the Bernoulli measure with parameter p.

3. PROOFS

By going to the dual process[5] we obtain

$$\mu_{L-1}(\tilde{x}) = \sum_{k=0}^{n}\binom{n}{k}P(\tilde{x};k;L)p(1)^k p(-1)^{n-k} \quad (3.1a)$$

$$\tilde{x} = (x_1,\ldots,x_n), \qquad -L+1 < x_i < L-1,$$
$$i = 1,\ldots,n, \quad x_1 < \cdots < x_n \quad (3.1b)$$

$$\mu_{L-1}(\tilde{x}) = \mu_{L-1}(\{\eta(xi) = 1, i = 1,\cdots,n\}) \quad (3.1c)$$

$P(\tilde{x};k;L)$ denotes the probability that k particles reach L before $-L$ and that $n-k$ reach $-L$ before L. The probability is taken with respect to the simple exclusion process for n particles starting at $\tilde{x}$ with the condition that once a particle is at $\pm L$ it disppears. It is known[6] that

$$P(\tilde{x};n;L) = \prod_{i=1}^{n}\left(\frac{L+x_i-i+1}{2L-i+1}\right) \quad (3.2a)$$

$$P(\tilde{x};0;L) = \prod_{i=1}^{n}\left[\frac{L-(x_i+n-i)}{2L-(n-i)}\right] \quad (3.2b)$$

Equation (3.2) does not determine $\mu_{L-1}(\tilde{x})$ except for the case $n = 2$. We will extend Eq. (3.2) to all n but only up to first order in $1/L$. There are special symmetry considerations which allow an exact computation for the case when all particles exit from the same side and we have not been able to find analogous arguments in general.

The usual technique in the estimation of probabilities for the simple exclusion process is based on the comparison with the free (independent) process; cf. Refs. 1 and 4, for instance. Basically one introduces a coupled process and typically the interacting and the corresponding free particles are at mutual distance within $\sqrt{t}$ (at time t). This was enough for the cases treated in Ref. 1; in the present problem the "error" could be catastrophic, even a shift by 1 between corresponding particles in the coupled process could determine a different exit for a particle in the two processes. This might occur with small probability, and in Ref. 2 we exploited this by reducing the problem to a two-particle case (the other particles behaving "normally") and then using Eq. (3.2). The proof unfortunately contained a mistake which we have not been able to fix using that line of approach. The main consideration in the present approach is that the probability $P(\tilde{x}; k; L)$ depends only on the paths of the particles; it does not matter what time is taken for each one to travel along its trajectory. We exploit this by introducing a "weak coupling" between the interacting and free processes; we lose the time correlation between the displacements of the corresponding particles but we can get much more accuracy on their space paths. To accomplish this we will introduce a "coordinate reduction," which is a typical tool to reduce pure hard-core interactions to free cases, namely, we change the position x_i of the interacting particle to $x_i' = x_i - i + 1$. In these coordinates all displacements are allowed, the actual position being recovered with the knowledge of the mutual ordering among the particles, [this is the reason for the appearance of $x_i - i + 1$ in Eq. (3.2a)]. The difficulty comes from the fact that times cannot be matched: when two interacting particles are at nearest-neighbor (n.n.) sites they separate more slowly than two free particles standing on the same site. We will show that the changes in the x_i' are those of an independent process, the times and the order at which they occur being, however, different. We will prove that there is an isomorphic mapping between the probability spaces on which the two processes are realized for which the space trajectories of each x_i' are mapped in the same space trajectory of the corresponding free particle; we agree to call this a weak coupling between the two processes.

We divide the remainder of the section into two parts: in the first one we describe the weak coupling, in the second one we use it to prove of Theorem 2.1.

3.1. Weak Coupling

We will introduce a probability space $(\Omega, \mathcal{F}, P)$ where both the independent and the interacting processes are realized. The space Ω is equipped with an increasing family of σ algebras $\mathcal{F}(t)$. The interacting process $\{x_1(t), \ldots, x_n(t)\}$ starts from $x_1, \ldots, x_n$, $(x_1 < x_2 < \cdots < x_n)$ and is measurable with respect to $\mathcal{F}(t)$. The independent process $x_1^0(t), \ldots, x_n^0(t)$ starts from $x_1', \ldots, x_n'$, $x_i' = x_i - i + 1$, $i = 1, \ldots, n$ and is anticipated with respect to the filtration $\mathcal{F}(t)$. More precisely, we will prove that for all $i = 1, \ldots, n$ and $t \in \mathbb{R}_+$ there are stopping times $T(t,i)$ such that $[x_i^0(t')]_{t' \in [0,t]}$ is measurable with respect to $\mathcal{F}_{T(t,i)}$.

The main result is the following:

Theorem 3.1. (a) $T(t,i)$ is nondecreasing in t, $T(t,i) \geqslant t$, and it is almost surely finite.

(b) There is a labeling rule for the simple exclusion process so that the following holds. Define for $y_1, \ldots, y_n$ $(y_i \neq y_j,\ i \neq j)$

$$k(i; y_1, \ldots, y_n) = \text{Cardinality}(j : y_j \leqslant y_i)$$

and given $y_1, \ldots, y_n$ as above,

$$y_i' = y_i - k(i; y_1, \ldots, y_n) + 1$$

we have that for all t

$$x_i'\big(T(t,i)\big) = x_i^0(t), \qquad i = 1, \ldots, n$$

As a consequence the sequence of jumpts occurring in the path of each independent particle are just the same as those of the corresponding interacting one in its "reduced coordinate."

(c) For every $e > 0$ there exist $A, B > 0$ so that

$$P\big(\big\{T(t,i) - t > t^{1/2+\epsilon}\big\}\big) \leqslant Ae^{-Bt^{\epsilon}}$$

$$P\big(\big\{|x_i(t) - x_i^0(t)| > t^{1/4+\epsilon}\big\}\big) \leqslant Ae^{-Bt^{\epsilon}}$$

Our first step is the explicit construction of the probability space $(\Omega, \mathcal{F}, P)$. For every $x \in \mathbb{Z}$ we introduce a Poison point process of parameter 1 and to each point a ± 1 mark is attached with independent symmetric probability. Furthermore, for every $2 \leqslant m \leqslant n$ we define a Bernoulli scheme with values on the permutation of $1, \ldots, m$, each permutation having the same probability. The space $(\Omega, \mathcal{F}, P)$ is the direct product of all these spaces.

The Filtration $\mathcal{F}(t)$. We consider the initial position of the particles $\tilde{x} = x_1, \ldots, x_n$ as fixed. The first stopping time τ_1 is the first time a mark appears among the sites $\tilde{x}$. For $t < \tau_1$ we put $\tilde{x}(t) = \tilde{x}$, τ_1 determines a site and a possible jump for the particle at that site: namely, if x_i is the site where the mark occurs and σ its value, then at time τ_1 the new configuration is $\tilde{x}' = x_1', \ldots, x_n'$ with $x_j' = x_j, j \neq i$, and $x_i' = x_i + \sigma$ if $x_i + \sigma \neq x_j$, $\forall j \neq i$, and $= x_i$ otherwise. Starting from $\tilde{x}(\tau_1)$ a second stopping time τ_2 and a trajectory $\tilde{x}(t)$, $\tau_1 \leqslant t \leqslant \tau_2$ are defined as before. Recursively are then defined $\tau_3, \ldots, \tau_n, \ldots$ and $\tilde{x}(t)$ up to $\tau_n \ldots$. We will now introduce a labeling of the particles; the corresponding process will be hereafter referred to as the labeled process. We say that a cluster occurs in the configuration $\tilde{y} = y_1, \ldots, y_n$ $(y_i \neq y_j \ i \neq j)$ if there are $m \geqslant 2$ nearest-neighbor particles. Particles keep their labels until they get into a cluster, at which time labels are uniformly randomized inside the cluster. This procedure starts at time zero. Assume there are k clusters in $\tilde{x}$, assign an arbitrary ordering among them and say there are $m_1, \ldots, m_k$ particles, respectively. Then the labels of the particles in the first cluster are changed according to the first permutation of m_1 integers appearing in Ω. Let it be $\pi(1), \ldots, \pi(m_1)$; then the label of the first particle in the cluster becomes that of the $\pi(1)$th and so on. For the second cluster we use the first permutation with m_2 integers if $m_2 \neq m_1$ or the second with m_1 if $m_2 = m_1$, and so on for all the others clusters. This defines the labeled configuration at time 0^+. The labels are then kept by the particles up to τ_m, the first among $\tau_1, \ldots, \tau_k, \ldots$ for which either a new cluster appears with respect to those in τ_{m-1} or an existing one increases. At that time the labels of the particles in that cluster are changed according to the first not already used permutation of m integers, if the cluster has m particles. $\mathcal{F}(t)$ is the smallest σ algebra in $\mathcal{F}$ for which $\mathbf{x}(t')$, $t' \leqslant t$, $\tau_i \wedge t$, $i \in \mathbb{N}$, and the sites and values of the marks corresponding to $\tau_i \wedge t$, $i \in \mathbb{N}$, are measurables.

The Interacting Process. It is easy to see that the process $\tilde{x}(t)$ (neglecting the labels of the particles) is the simple exclusion process starting at $\tilde{x}$.

The independent process $\mathbf{x}^0(t)$ has initial position $\mathbf{x}', x_i' = x_i - i + 1$, where $\mathbf{x}$ is the labeled configuration of the interacting process at time 0^+. The times when particles move are $\tau_1, \tau_2, \ldots$. The particle which moves at time τ_m has the label of the interacting particle that at time τ_{m-1} is at the site which determines τ_m. To specify the jump of the independent particle we do the following. Given the labeled configuration $\mathbf{x}$ we define $\mathbf{x}'$ as $x_i' = x_i - k(i,x) + 1$, $i = 1, \ldots, n$. Let $\delta x_i'(1), \ldots, \delta x_i'(m), \ldots$, $i = 1, \ldots, n$ be the changes occurring in $x_i'(t)$, $t \in \mathbb{R}_+$. We then say that the changes $\delta x_i^0(1), \ldots, \delta x_i^0(m), \ldots, i = 1, \ldots, n$ are the same as $\delta x_i'(1), \ldots, \delta x_i'(m), \ldots$, respectively, $i = 1, \ldots, n$.

It is easy to verify that the times τ_m are distributed according to a Poisson point process of intensity n, that the label of the independent particle which moves at time τ_m has probability $1/n$ independently of $\mathcal{F}(t')$, $t' < \tau_m$, and that the random variables $\delta x_i'(m)$, $i \in \{1, \ldots, n\}$, $m \in \mathbb{N}$ are independent symmetric random variables, so that $\mathbf{x}^0(t)$ describes an independent process for n particles starting at $\mathbf{x}$.

Given $i \in \{1, \ldots, n\}$ and $t \in \mathbb{R}_+$ the stopping time $T(t,i)$ is the greatest between t and the time at which $\delta x_i'(m)$ occurs, where m is the $[\mathcal{F}(t)$ measurable$]$ number of times x_i^0 moves up to time t. $x_i^0(t')$, $0 \leqslant t' \leqslant t$ is then $\mathcal{F}_{T(t,i)}$ measurable.

Parts (a) and (b) of Theorem 3.1 are direct consequences of the above construction. The difference between $T(t,i)$ and t depends on how many times before t a mark appeared which referred to the i particle and for which the x_i' coordinate did not change. This can be estimated in terms of the time any two interacting particles are close (n.n.) which increases less than $t^{1/2+\epsilon}(\epsilon > 0)$ with probability greater than $1 - A'e^{-A''t^\epsilon}$, $A'' > 0$ (A' and A'' depending on ϵ); cf. Ref. 4. It is then easy to get the estimates of Theorem 3.1(c).

We conclude this subsection with the following remark. Let $(\mathfrak{X}, \mathcal{F}, P)$ and $(\mathfrak{X}^0, \mathcal{F}^0, P^0)$ be probability spaces where the interacting labeled process $\mathbf{x}(t)$ and $\mathbf{x}^0(t)$ are realized with $\mathcal{F}[\mathcal{F}^0]$ being the σ algebra generated by $\{\mathbf{x}(t)\}[\{\mathbf{x}^0(t)\}]$. Then there exists an isomorphic mapping ψ from $(\mathfrak{X}, \mathcal{F}, P)$ to $(\mathfrak{X}^0, \mathcal{F}^0, P^0)$, which makes $\delta x_i'(m) = \delta x_i^0(m)$ for all $i \in \{1, \ldots, n\}$ and $m \in \mathbb{N}$. We have already proven in Theorem 3.1 that the process $\delta x_i'(m)$, $\delta x_i^0(m)$ have the same distribution; therefore an isomorphic correspondence between the atoms of the measurable partition π generated by the $\delta x_i'(m)$ variables in $\mathfrak{X}$ and π^0 generated by $\delta x_i^0(m)$ in $\mathfrak{X}^0$ is set. The conditional probability of P to each atom of π is nonatomic and the same happens for π^0. Furthermore, both P and P^0 are nonatomic after relativitation to the σ algebras generated by π and π^0, respectively. By the theorem of Ref. 7 π has an orthogonal complement, namely, there is $\pi^\perp$ such that P is the direct product of P relativized to $\mathfrak{X}/\pi$ and $\mathfrak{X}/\pi^\perp$. Analogously for π^0. Both P on $\mathfrak{X}/\pi^\perp$ and P^0 on $\mathfrak{X}^0/\pi^{0\perp}$ are nonatomic Lebesgue measures, hence isomorphic for a mapping Φ. The above-defined correspondence between the atoms of π and π^0 together with Φ defines the isomorphic mapping ψ between $(\mathfrak{X}, \mathcal{F}, P)$ and $(\mathfrak{X}^0 \mathcal{F}^0, P^0)$.

3.2. Proof of Theorem 2.1.

We use the labeling described in Section 3.1 to rewrite Eq. (3.3) as

$$\mu_{L-1}(\tilde{x}) = \sum_{\substack{\epsilon_1 \ldots \epsilon_n \\ \epsilon_i = \pm 1}} P(\mathbf{x}; \boldsymbol{\epsilon}; L) \prod_{i=1}^{n} p(\epsilon_i) \tag{3.3}$$

$P(\mathbf{x}; \boldsymbol{\epsilon}; L)$ is the probability that particle i reaches $\epsilon_i L$ before $-\epsilon_i L$. The probability is computed with resepct to the labeled exclusion process as defined in Section 3.1 with the rule that once a particle reaches $\pm L$ it disappears (the others moving according to the same prescriptions but with reference to a number of particles decreased by one). We have the following:

Lemma 3.1. For $\xi \in (-1, 1)$,

$$\lim L\left\{\left[\mu_L(\eta(x_i + [\xi L]) = 1, i = 1, \dots, n) - p(\xi_L)^n\right] - \left[\sum_{\epsilon_1 \dots \epsilon_n}\{P(\mathbf{x} + [\xi L]; \boldsymbol{\epsilon}; L+1) - P^0(\mathbf{x} + [\xi L]; \boldsymbol{\epsilon}; L+1)\}\prod_{i=1}^n p(\epsilon_i)\right]\right\}$$

$$= \sum_{i=1}^n \frac{1}{2}[p(1) - p(-1)]x_i p(\xi)^{n-1}$$

$$\xi_L = (L+1)^{-1}[\xi L], \qquad \mathbf{x} + [\xi L] \equiv (x_1 + [\xi L], \dots, x_n + [\xi L])$$

$p(\xi)$ is defined in Eq. (2.2) and $P^0(\mathbf{x}; \boldsymbol{\epsilon}; L)$ is the probability that particle i reaches $\epsilon_i L$ before $-\epsilon_i L$ for the independent process starting at $\mathbf{x}$.

Proof. It is easy to see that $P^0(x; \epsilon; L) = (L + \epsilon x)/2L$ and from this the lemma follows easily. ■

The above estimate leads to the appearance of the one-body potential in Theorem 2.1.

From Lemma 3.1 we reduce the problem to the computation of the limit of $L[P(\mathbf{x} + [\xi L]; \boldsymbol{\epsilon}; L) - P^0(\mathbf{x} + [\xi L]; \boldsymbol{\epsilon}; L)]$. The coupling we have introduced in Section 3.1 compares the labeled simple exclusion process starting at $\mathbf{x}$ with the independent one starting at $\mathbf{x}'$. It is therefore necessary to estimate the contribution of the independent process when it starts from $\mathbf{x}$ and from $\mathbf{x}'$. We easily have (and therefore we omit its proof) the following:

Lemma 3.2. For $\xi \in (-1, 1)$,

$$\lim L \sum_{\epsilon_1 \dots \epsilon_n}\left[P^0(\mathbf{x} + [\xi L]; \boldsymbol{\epsilon}; L) - P^0(\mathbf{x}' + [\xi L]; \boldsymbol{\epsilon}; L)\right]\prod_{i=1}^n p(\epsilon_i)$$

$$= p(\xi)^{n-2}\sum_{\substack{(i,k) \\ i \neq k}}\sum_{\epsilon_1\epsilon_2}\frac{1}{4}(1 + \epsilon_1\xi)\epsilon_2 p(\epsilon_1)p(\epsilon_2)$$

We are now left with the problem of comparing $P(\mathbf{x} + [\xi L]; \boldsymbol{\epsilon}; L)$ and $P^0(\mathbf{x}' + [\xi L]; \boldsymbol{\epsilon}; L)$. For notational simplicity we only consider $\xi = 0$. We realize the independent and simple exclusion processes on $(\Omega, \mathcal{F}, P)$ as in Section 3.1. However, it should be remembered that once an interacting particle reaches $\pm L$ it disappears and so at later times the construction of the labeled interacting process takes account only of the remaining particles. Memory of the disappeared particle is, however, left when one recovers the positions of the particles from their reduced coordinate description: in the ordering of the configuration the particles which reached $-L$ are the first ones, those at $+L$ the last ones. (For the independent process, of course, the above remarks are irrelevant.)

We introduce the stopping time $\lambda_i(+)[\lambda_i(-)]$ as the first time $x_i^0(t)$ reaches $L-n[-L+1]$ and $\lambda_i = \min\{\lambda_i(+), \lambda_i(-)\}$, $i = 1, \ldots, n$. For $i = 1, \ldots, n$ let

$$\begin{aligned} A_i(j,-) &= \{\lambda_i = \lambda_i(-), x_i^0(+) > -L-j, \forall t \leqslant \lambda_i(+)\} \\ A_i(j,+) &= \{\lambda_i = \lambda_i(+), x_i^0(+) < L-n+j+1, \forall t \leqslant \lambda_i(-)\} \\ A_i(j) &= A_i(j,-) \cup A_i(j,+) \\ A^c &= \left[\bigcup_{i=1}^{n} A_i(n-1)\right]^c \end{aligned} \tag{3.4}$$

By Theorem 3.1(b) we have that ($T_i[T_i^0]$ below are the stopping times at $\pm L$ for the interacting [independent] particle i)

$$\begin{aligned} &\int_{A^c} dP\, 1(\{x_i(T_i) = \epsilon_i L, i = 1, \ldots, n\}) \\ &\qquad = \int_{A^c} dP\, 1\left(\left\{x_i^0\left(T_i^0\right) = \epsilon_i L, i = 1, \ldots, n\right\}\right) \end{aligned} \tag{3.5}$$

We choose $\delta > 0$ small enough, for instance $\delta = 1/100$, and we set $T(L) = L^\delta$. We then define for $i = 1, \ldots, n$

$$\begin{aligned} B_i = &\{\lambda_i = \lambda_i(-), \exists t < \lambda_i(+), t < \lambda_i + T(L), x_i^0(t) = -L-n+1\} \\ &\sqcup \{\lambda_i = \lambda_i(+), \exists t < \lambda_i(-), t < \lambda_i + T(L), x_i^0(t) = L\} \end{aligned} \tag{3.6}$$

$$\begin{aligned} D_i = &\{\lambda_i = \lambda_i(-), x_i^0(t) \geqslant -L+1, \forall t \in [\lambda_i + T(L), \lambda_i(+)]\} \\ &\sqcup \{\lambda_i = \lambda_i(+), x_i^0(t) \leqslant L-n, \forall t \in [\lambda_i + T(L), \lambda_i(-)]\} \end{aligned} \tag{3.7}$$

$$\begin{aligned} C = &\{L^{2-\delta} \leqslant \lambda_i \leqslant L^{2+\delta}, i = 1, \ldots, n; \\ &|\lambda_i - \lambda_j| > L^{2-\delta}\ \forall i \neq j;\ T(i,t) - t < t^{1/2+\delta}\ \forall t > L^{2-\delta}\} \end{aligned} \tag{3.8}$$

It remains to compare the interacting and independent processes in the set $\sqcup_{i=1}^n A_i(n-1)$. In order to write the exit condition for the interacting particles in a computable way we restrict this space by imposing further conditions. We therefore need the following:

Lemma 3.3. For every $i \in \{1, \ldots, n\}$,

$$\lim LP\left[A_i \cap \left(C \cap D_i \bigcap_{j \neq i} B_j\right)^c\right] = 0$$

where $A_i = A_i(n-1)$.

Proof. We use the following[3] for the distribution density $F(t,a)$ of the stopping time at a for a simple random walk which starts at the origin: as a diverges

$$F(t,a) \sim \text{const}|a| t^{-3/2} e^{-a^2/2t} \tag{3.9}$$

From this it easily follows that

$$\lim P\left(L^{2-\delta} \leqslant \lambda_i \leqslant L^{2+\delta}\right) = 1, \qquad i = 1, \ldots, n \tag{3.10a}$$

$$\lim P\left(|\lambda_i(\epsilon) - \lambda_j(\epsilon')| > L^{2-\delta}\right) = 1, \qquad \forall \epsilon, \epsilon' = \pm 1; \forall i \neq j \tag{3.10b}$$

By use of Eq. (3.10) and Theorem 3.1(c) we have that

$$\lim LP(A_i \cap C^c) = 0$$

For $j \neq i$

$$P(A_i \cap B_j^c) = P(A_i)P(B_j^c)$$

and $P(A_i)$ behaves as L^{-1} while using Eq. (3.9) $\lim_{L\to\infty} P(B_j^c) = 0$. We finally have

$$P\left[A_i(n-1,-) \cap D_i^c\right] \leqslant P(\mathcal{J}^0 \cap \mathcal{J})$$

$$\begin{aligned}
\mathcal{J}^0 &= \left\{x_i^0(t) > -L - n + 1, \forall t \in \left[\lambda_i(-), \lambda_i(-) + T(L)\right]\right\} \\
&\quad \sqcup \left\{\lambda_i(-) + T(L) \geqslant \lambda_i(+)\right\} \\
\mathcal{J} &= \left\{\exists \hat{\tau} \in \left[\lambda_i(-) + T(L), \lambda_i(+)\right] : -L - n + 1 < x_i^0(\hat{\tau}) \leqslant -L;\right. \\
&\quad \left. x_i^0(t) > -L - n + 1, \forall t \in \left[\hat{\tau}, \lambda_i(+)\right]\right\}
\end{aligned}$$

$$P(A_i(n-1,-) \cap D_i^c) \leqslant \mathbf{E}\left(1(\mathcal{J}_0)\mathbf{E}\left(1(\mathcal{J})x_i^0(t), \forall t \in (0,\hat{\tau}]\right)\right) \leqslant HL^{-1}P(\mathcal{J}^0)$$

where H is a suitable constant independent of L. By Eq. (3.9), $\lim_{L\to\infty} P(\mathcal{J}^0) = 0$. Analogous estimate holds for $A_i(n-1,+)$ and then the lemma follows easily. ■

Lemma 3.4. Let $T_i[T_i^0]$, $i = 1, \ldots, n$, be the stopping time at $\pm L$ for $x_i(t)[x_i^0(t)]$. We have that for $i = 1, \ldots, n$,

$$\begin{aligned}
\lim L\mathbf{E}\Bigg(& 1(A_i) \prod_{j=1}^{n} 1(x_j(T_j) = \epsilon_j L) - \prod_{j \neq 1} 1(\lambda_j = \lambda_j(\epsilon_j)) \\
& \times \Bigg\{ 1 + \epsilon_i \Bigg[\Bigg(1(\lambda_i = \lambda_i(-)) 1\Big(A_i\Big(\sum_{k\neq i} \varphi_i(k), -\Big)\Big) \\
& \qquad - 1(\lambda_i = \lambda_i(+)) 1\Big(A_i\Big(\sum_{k\neq i} \psi_i(k), +\Big)\Big)\Bigg)\Bigg]\Bigg\}\Bigg) = 0
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}\varphi_i(k) &= 1 \qquad \text{if } \lambda_i = \lambda_i(-), \lambda_k = \lambda_k(-) < \lambda_i(-) \\ &= 0 \qquad \text{otherwise}\end{aligned} \tag{3.12}$$

$$\begin{aligned}\psi_i(k) &= 1 \qquad \text{if } \lambda_i = \lambda_i(+) \text{ and } \lambda_k = \lambda_k(+) < \lambda_i(+) \\ &= 0 \qquad \text{otherwise}\end{aligned} \tag{3.13}$$

Proof. By Lemma 3.3 we have

$$\lim L\left[P\left(A_i \bigcap_{j=1}^{n} \{x_j(T_j) = \epsilon_j L\}\right) - P\left(A_i \bigcap_{j=1}^{n} \{x_j(T_j) = \epsilon_j L\} \cap C \cap D_i \bigcap_{j\neq i} B_j\right)\right] = 0 \tag{3.14}$$

By definition of B_j we have

$$B_j \cap \{x_j(T_j) = \epsilon_j L\} = \{\lambda_j = \lambda_j(\epsilon_j)\} \cap B_j$$

For trajectories in $\bigcap_{j\neq i} B_j \cap C$ we have that if $\lambda_i = \lambda_i(-)$ at that time $\sum_{k\neq i}\varphi_i(k)$ particles have already reached $-L$ for L large enough. Because we consider trajectories in $D_i \cap C$ we also have that the exit condition at $-L$ for particle i reads as the condition for $x_i^0(t)$ to reach $-L - \sum_{k\neq i}\varphi_i(k)$. Analogous argument is used when $\lambda_i = \lambda_i(+)$. By Lemma 3.3 we complete the proof of Eq. (3.11). ■

Let T_i^0 be the stopping time at $\pm L$ for $x_i^0(t)$, we then have

$$\begin{aligned}\lim L\Big[& P(\{x_j(T_j) = \epsilon_j L, j = 1, \ldots, n\}) - P\left(\left\{x_j^0\left(T_j^0\right) = \epsilon_j L, j = 1, \ldots, n\right\}\right)\Big] \\ = \lim L \sum_{i=1}^{n} \mathbf{E}\Bigg(& 1(\lambda_j = \lambda_j(\epsilon_j) \forall j \neq i)\epsilon_i \\ & \times \Bigg\{\left[1\left(A_i\left(\sum_{k\neq i}\varphi_i(k) + 1, -\right)\right) - 1(A_i(1, -))\right] 1(\lambda_i = \lambda_i(-)) \\ & + \left[1(A_i(n, +)) - 1\left(A_i\left(\sum_{k\neq i}\psi_i(k) + 1, +\right)\right)\right] \\ & \times 1(\lambda_i = \lambda_i(+))\Bigg\}\Bigg)\end{aligned} \tag{3.15}$$

After conditioning to $x_i^0(t)$ for $t \leqslant \lambda_i$ and to $x_j^0(t)$, $\forall t$, $\forall j \neq i$, we get that

$$\begin{aligned}\lim L\Big[& P(\{x_j(T_j) = \epsilon_j L, j = 1, \ldots, n\}) \\ & - P\big(\{x_j^0(T_j^0) = \epsilon_j L, j = 1, \ldots, n\}\big)\Big] \\ = & \lim_{L\to\infty} \sum_{i=1}^{n} \sum_{k\neq i} \frac{1}{2} P(\lambda_j = \lambda_j(\epsilon_j), \forall j \neq i, k) \\ & \times \mathbf{E}(\epsilon_i 1(\lambda_k = \lambda_k(\epsilon_k))\{\varphi_i(k)1(\lambda_i = \lambda_i(-)) \\ & + (1 - \psi_i(k))1(\lambda_i = \lambda_i(+))\}) \end{aligned} \tag{3.16}$$

Equation (3.16) reduces the problem to the computation of exit probabilities for independent particles. It is easy to see that the expectation on the right-hand side of Eq. (3.16) in the limit $L \to \infty$ is

$$\tfrac{1}{8} \text{ if } \epsilon_i = \epsilon_k = 1 \tag{3.16a}$$

$$-\tfrac{3}{8} \text{ if } \epsilon_i = \epsilon_k = -1 \tag{3.16b}$$

$$\tfrac{1}{4} \text{ if } \{\epsilon_i = 1, \epsilon_k = -1, \epsilon_i = -1, \epsilon_k = 1\} \tag{3.16c}$$

By Lemmas 3.1, 3.2, and Eqs. (3.15), (3.16), we obtain the first-order correction to $\mu_L(\eta(x_i) = 1, i = 1, \ldots, n)$ and then it is easy to check that this agrees with the estimate of Theorem 2.1 at $\xi = 0$. The case $\xi \neq 0$ is completely analogous.

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NOTE ADDED IN PROOF

We are indebted to Herbert Spohn for the following remark. If one considers the fluctuation field

$$\xi^L(f) = \frac{1}{\sqrt{L}} \sum_{x=-L}^{L} f\left(\frac{x}{L}\right)[\eta_x - \mu_L(\eta_x)]$$

in the state μ_L, then one can show that

$$\lim_{L\to\infty} \mu_L(\xi^L(f)\xi^L(g)) = \int dx\, dy\, f(x)g(y)\langle\xi(x)\xi(y)\rangle$$

The covariance of the limit fluctuation field is given by

$$\langle\xi(x)\xi(y)\rangle = p(x)[1 - p(x)]\delta(x-y) + (p')^2\Delta^{-1}(x,y), \quad |x| \leqslant 1, |y| \leqslant 1$$

where $\Delta^{-1}(x, y)$ is the kernel of the inverse Laplacian on $[-1, 1]$ with

Dirichlet boundary conditions. At coinciding arguments

$$\Delta^{-1}(x,x) = \tfrac{1}{2}(1 - x^2)$$

Therefore also in the steady state one finds the structure obtained for time-dependent states: the strength of the two-body interaction is determined by the regular part of the covariance of the fluctuation field at coinciding arguments.

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